# Abel, Gauss and related transforms for electromagnetic applications

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***Abstract*** — **The definition of the Gauss transform is based on the eigenstructure of the Abel transform. This allows to find the inverse Gauss transforms of the classical isotropic Green's functions in any dimension. This has consequences for the calculation of certain pertinent integrals in electromagnetics in terms of the related finite and discrete Abel transforms.**

## 1 Introduction

The Abel transform, known for its relationship with the Radon transform [1], is a well-known tool in computerized tomography and image processing. It is also known for its relationship with the Bessel-Hankel, Struve and Y-transforms [2]. It is much less known, and most important for electromagnetic applications, that the Abel transform, when applied to the isotropic frequency-domain Green's functions of dimension $n$, results in the isotropic frequency-domain Green's function of dimension $n-1$. From the fact [3] that the Abel transform admits zero-mean Gaussian distributions with different (possibly complex) variances as eigenfunctions, it is then possible to define the Gauss transform and find the inverse Gauss transforms of the Green's functions in any dimension. This has consequences for the calculation of certain pertinent integrals in electromagnetics in terms of the related finite and discrete Abel transforms.

## 2 Abel, Gauss and related transforms

The (infinite) Abel transform and its inverse for functions defined over $R_+$ are given by [1, 2]

$$\Phi(r) = \mathcal{A}f \equiv 2\int_r^\infty \frac{f(t)t}{\sqrt{t^2-r^2}}\,dt \quad (1)$$

$$f(r) = \mathcal{A}^{-1}\Phi \equiv -\frac{1}{\pi}\int_r^\infty \frac{\Phi'(t)}{\sqrt{t^2-r^2}}\,dt \quad (2)$$

This can also be written as

$$\Phi(r) = 2\int_0^\infty f\left(\sqrt{r^2+x^2}\right)dx \quad (3)$$

$$f(r) = -\frac{1}{\pi}\int_0^\infty \frac{\Phi'\left(\sqrt{r^2+x^2}\right)}{\sqrt{r^2+x^2}}\,dx \quad (4)$$

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An interesting property of the Abel transform is that its square is a simple integral operator, i.e., we have

$$\xi(r) = \mathcal{A}^2 f = 2\pi\int_r^\infty f(t)t\,dt \quad (5)$$

$$f(r) = -\frac{\xi'(r)}{2\pi r} \quad (6)$$

From (3) it is seen that the Abel transform admits the continuous eigenspectrum of Gaussians, i.e.,

$$\mathcal{A}\,e^{-\gamma r^2} = \sqrt{\frac{\pi}{\gamma}}\,e^{-\gamma r^2} \qquad \Re\gamma > 0 \quad (7)$$

Most important for electromagnetic applications, the Abel transform applied to the isotropic frequency-domain Green's functions ($e^{i\omega t}$ time dependence assumed)

$$g_1(r;k) = \frac{i}{2k}\,e^{-ikr}, \quad g_2(r;k) = \frac{i}{4}\,H_0^{(2)}(kr), \qquad g_3(r;k) = -\frac{1}{4\pi r}\,e^{-ikr} \quad (8)$$

where $\Im k < 0$, say $k = -i\epsilon + \omega/c$ ($\epsilon > 0$), is such that $\mathcal{A}g_n = g_{n-1}$. The link with the cosine, sine, Bessel-Hankel and Struve transforms [4] p. 74 is given by :

$$\mathcal{C}\Phi \equiv \int_0^\infty \cos(\rho r)\,\Phi(\rho)\,d\rho = \pi\int_0^\infty J_0(\rho r)f(\rho)\rho\,d\rho \equiv \pi\mathcal{B}f \quad (9)$$

$$\mathcal{S}\Phi \equiv \int_0^\infty \sin(\rho r)\,\Phi(\rho)\,d\rho = \pi\int_0^\infty \mathbf{H}_0(\rho r)f(\rho)\rho\,d\rho \equiv \pi\mathcal{H}f \quad (10)$$

where $\mathbf{H}_0(\cdot)$ is the Struve function. Note that (9)-(10) can be formally written as $\mathcal{C}\mathcal{A} = \pi\mathcal{B}$ and $\mathcal{S}\mathcal{A} = \pi\mathcal{H}$. The inverse of the Bessel-Hankel transform is itself, while the inverse of the Struve or $H-$transform is the $Y-$transform

$$\mathcal{Y}f \equiv \int_0^\infty Y_0(\rho r)f(\rho)\rho\,d\rho \quad (11)$$

where $Y_0(\cdot)$ is the Bessel-Neumann function.
There is an important relationship with the Gauss

transform, which we define for fixed $\alpha$, $\Re\alpha > 0$ as

$$f(r) = \mathcal{G}_\alpha h \equiv \int_0^\infty e^{-\pi\alpha r^2/\rho^2} h(\rho)\, d\rho \tag{12}$$

and the related transform

$$\hat{f}(r) = \hat{\mathcal{G}}_\alpha h \equiv \int_0^\infty e^{-r^2\rho^2/4\pi\alpha} h(\rho)\, d\rho \tag{13}$$

Note that $\hat{\mathcal{G}}_\alpha$ can be obtained from $\mathcal{G}_\alpha$ by means of

$$\hat{\mathcal{G}}_\alpha h = \mathcal{G}_{1/\alpha} \hat{h} \tag{14}$$

where $\hat{h}$ is the $L_1[R_+]$ isometry

$$\hat{h}(\rho) = \mathcal{T}_1 h \equiv h\left(\frac{2\pi}{\rho}\right) \frac{2\pi}{\rho^2} \tag{15}$$

To obtain the inverse Gauss transform we take the Gauss transform of $f(r)$, yielding

$$\begin{aligned} \mathcal{G}_\alpha f &= \frac{1}{2} \int_0^\infty e^{-2\pi\alpha r/\rho} \frac{\rho}{\sqrt{\alpha}} h(\rho)\, d\rho \\ &= \frac{1}{2} \int_0^\infty e^{-\alpha r t}\, \mathcal{T}_1 \left[\frac{t}{\sqrt{\alpha}} h(t)\right] dt \end{aligned} \tag{16}$$

which, by putting $s = \alpha r$, allows to retrieve $\mathcal{T}_1[t h(t)/\sqrt{\alpha}]$ and hence $h(t)$ by means of the inverse Laplace transform. With respect to the derivatives of $f(r)$ there are two important formulas that can be proved straightforwardly, namely

$$\frac{f'(r)}{r} = -2\pi\alpha\, \mathcal{G}_\alpha \left[\frac{h(\rho)}{\rho^2}\right] \tag{17}$$

$$f''(r) = -2\pi\alpha\, \mathcal{G}_\alpha \left[\frac{h'(\rho)}{\rho}\right] \tag{18}$$

It is not too hard to show that the Abel, Bessel-Hankel and cosine transforms can be written in the Gauss domain as

$$\mathcal{A} f = \mathcal{G}_\alpha \left[\frac{\rho}{\sqrt{\alpha}} h(\rho)\right] \tag{19}$$

$$\mathcal{B} f = \frac{1}{2\pi} \hat{\mathcal{G}}_\alpha \left[\frac{\rho^2}{\alpha} h(\rho)\right] \tag{20}$$

$$\mathcal{C} f = \frac{1}{2} \hat{\mathcal{G}}_\alpha \left[\frac{\rho}{\sqrt{\alpha}} h(\rho)\right] \tag{21}$$

Again, from (19)-(21), we see that $\mathcal{C}\mathcal{A} = \pi\mathcal{B}$. The most interesting property of the Gauss transform, besides (19), is that it transforms (zero-mean) Gaussians into exponentials, i.e.,

$$\mathcal{G}_\alpha \left[e^{-\gamma\rho^2/4\pi}\right] = \frac{\pi}{\sqrt{\gamma}} e^{-\sqrt{\alpha\gamma} r} \qquad \Re\gamma > 0 \tag{22}$$

From (19) and (22) we readily obtain that the isotropic Green's functions $g_n(r;k)$ admit simple representations in the Gauss domain, i.e.,

$$g_n(r;k) = -\mathcal{G}_\alpha \left[\frac{e^{-\beta\rho^2/4\pi}}{2\pi\sqrt{\alpha}(\rho/\sqrt{\alpha})^{n-1}}\right] \tag{23}$$

where $ik = \sqrt{\alpha\beta}$ and $\Re\beta > 0$. For example we can take $\alpha = ik/|k|$ and $\beta = ik|k|$. Note that result (23) is compatible with (17)-(18) since the defining second-order differential equation for $g_n(r;k)$, i.e.,

$$f''(r) + (n-1)\frac{f'(r)}{r} + k^2 f(r) = 0 \tag{24}$$

for $r > 0$, is equivalent in the Gauss domain with the first-order differential equation

$$-2\pi\alpha \left[\frac{h'(\rho)}{\rho} + (n-1)\frac{h(\rho)}{\rho^2}\right] + k^2 h(\rho) = 0 \tag{25}$$

which exhibits the general solution

$$h(\rho) = C\, e^{-\beta\rho^2/4\pi}/\rho^{n-1} \tag{26}$$

## 3 Calculating the finite Abel transform

The finite Abel transform, defined as

$$\Psi(r,z) = \mathcal{A}_z f \equiv 2\int_0^z f\left(\sqrt{r^2+x^2}\right) dx \tag{27}$$

with $\Psi(r,\infty) = \Phi(r)$ and $z > 0$, while being simply inverted by means of $d\Psi(r,z)/dz = 2f(\sqrt{r^2+z^2})$, is not in general easily calculated. It can however be written as

$$\begin{aligned} \Psi(r,z) &= \Phi(r) \\ &+ \frac{2}{\pi}\int_z^\infty dx \int_0^\infty \frac{\Phi'\left(\sqrt{r^2+x^2+u^2}\right)}{\sqrt{r^2+x^2+u^2}} du \end{aligned} \tag{28}$$

which, after taking polar coordinates $x = p\cos\theta$, $u = p\sin\theta$, transforms to

$$\Psi(r,z) = \Phi(r) - \frac{2}{\pi}\int_0^{\pi/2} \Phi\left(\sqrt{r^2 + z^2/\cos^2\theta}\right) d\theta \tag{29}$$

We can also write down the finite Abel transform in the Gauss domain as

$$\Psi(r,z) = \mathcal{G}_\alpha \left[\frac{\rho}{\sqrt{\alpha}} \operatorname{erf}\left(\frac{\sqrt{\alpha\pi}z}{\rho}\right) h(\rho)\right] \tag{30}$$

Note that (29) and (30) are equivalent representations, since it is known that

$$\frac{2}{\pi}\int_0^{\pi/2} e^{-\pi\alpha z^2/\rho^2\cos^2\theta} d\theta = 1 - \operatorname{erf}\left(\frac{\sqrt{\alpha\pi}z}{\rho}\right) \tag{31}$$

This implies that

$$\mathcal{A}_z\, g_n(r;k) = g_{n-1}(r;k) - \frac{2}{\pi}\int_0^{\pi/2} g_{n-1}\left(\sqrt{r^2+z^2/\cos^2\theta};k\right)\,d\theta \quad (32)$$

or equivalently

$$\mathcal{A}_z\, g_n(r;k) = -\mathcal{G}_\alpha\left[\frac{e^{-\beta\rho^2/4\pi}}{2\pi\sqrt{\alpha}(\rho/\sqrt{\alpha})^{n-2}}\,\mathrm{erf}\left(\frac{\sqrt{\alpha}\pi z}{\rho}\right)\right] \quad (33)$$

For example, in the case $n=2$, equation (32) translates to

$$\int_0^z H_0^{(2)}\left(k\sqrt{r^2+x^2}\right)dx = \frac{1}{k}\left[e^{-ikr} - \frac{2}{\pi}\int_0^{\pi/2} e^{-ik\sqrt{r^2+z^2/\cos^2\theta}}d\theta\right] \quad (34)$$

One could argue that equation (34) means we have traded one integral for another, but this is not quite the case, since the left-hand integral requires function calls to the 'complicated' function $H_0^{(2)}(\cdot)$, while the right-hand integral requires only function calls to the 'simple' exponential function. If we take $k=1-i$, $r=1$ and $z=100$, the integral (34), calculated with the MATHEMATICA 5.1® function NIntegrate, is evaluated for both integrals (as it should be) as $0.254163-0.0553969i$, but the left-hand integral requires 0.841 CPU seconds while the right-hand integral requires only 0.01 CPU seconds to complete. If, with the same $k$ and $r$ values we evaluate the integrals for $z=1,2,\cdots,100$ we need 61.569 total CPU seconds for the left-hand integrals and only 0.621 total CPU seconds for the right-hand integrals, which is about a hundred times faster. The computations were performed on a x86 PC running Windows NT 4.0.

## 4 Calculating the discrete Abel transform

Since $\Phi(r)$ can be written as

$$\Phi(r) = \mathcal{A}\,f = \int_{-\infty}^{\infty} f\left(\sqrt{r^2+(z-t)^2}\right)dt \quad (35)$$

for $z\in R$, it is plausible that a discrete evaluation, in the Riemann sense, of the integral (35) could provide some pertinent approximation to $\Phi(r)$. This is embodied in the discrete Abel transform, which is also of importance in the study of periodic Green's functions, and is defined as

$$\Xi_\Delta(r,z) = \mathcal{A}_{z,\Delta}\,f \equiv \Delta\sum_{n\in Z} f\left(\sqrt{r^2+(z-n\Delta)^2}\right) \quad (36)$$

where for $\Delta=0$ we take $\Xi_0(r,z)=\Phi(r)$. It is seen that $\Xi_\Delta(r,z)$ is periodic in $z$ with period $\Delta$. In the Gauss domain, $\Xi_\Delta(r,z)$ can be written as

$$\Xi_\Delta(r,z) = \mathcal{G}_\alpha\left[\frac{\rho}{\sqrt{\alpha}}\,\theta_3\left(\frac{\pi z}{\Delta}, e^{-\pi\rho^2/\alpha\Delta^2}\right)h(\rho)\right] \quad (37)$$

This follows from the Poisson summation formula, see e.g. [5] p. 485

$$\frac{1}{\sqrt{\pi t}}\sum_{m\in Z} e^{-(m-x)^2/t} = \theta_3\left(\pi x, e^{-t\pi^2}\right) \qquad t>0 \quad (38)$$

where $\theta_3(\cdot)$ is the elliptic theta function defined as

$$\theta_3(u,q) = 1 + 2\sum_{m=1}^{\infty}\cos(2mu)\,q^{m^2} \quad (39)$$

Regarding the discrete Abel transform of the isotropic Green's functions, it is not too hard to show from formulas (23) and (38)-(39) that we have

$$\mathcal{A}_{z,\Delta}\,g_n(r;k) = g_{n-1}(r;k) + 2\sum_{m=1}^{\infty} g_{n-1}(r;k_m)\,\cos(2m\pi z/\Delta) \quad (40)$$

where

$$ik_m = \sqrt{\alpha\beta+(2\pi m/\Delta)^2} \quad (41)$$

For example, in the case $n=2$, equation (40) translates to

$$\Delta\sum_{n\in Z} H_0^{(2)}\left(k\sqrt{r^2+(z-n\Delta)^2}\right) = \frac{2e^{-ikr}}{k} + 4\sum_{m=1}^{\infty}\frac{e^{-ik_m r}}{k_m}\cos(2m\pi z/\Delta) \quad (42)$$

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